Estimation of the Fundamental Frequency of Beams and Plates with Varying Thickness

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I. Introduction

A perturbation method presented here provides an approximate technique for the solution of the free vibration of beams and plates with some general variation in thickness. The advantage of the perturbation method over the other approximate techniques, such as Rayleigh-Ritz, Galerkin, or finite element, is that once the solution is determined for a general variation of thickness then the effect of small changes in thickness variation can be determined without resolving the original free-vibration problem. The simplicity and speed of the perturbation method should make it a valuable tool to the aerospace structural designer for determining the vibrational characteristics of least-weight beams and plates. Chehil and Dua[1] have previously used a perturbation method to determine the buckling stress of plates with variable thickness.

II. Development of Perturbation Equations for Free Motion

A. Free Vibration of Beams

The differential equation describing free transverse motion of a beam (with variable thickness and unit width) is

$$\mathrm{d}^2/\mathrm{d}x[(Eh^3/12)(\mathrm{d}^2w/\mathrm{d}x^2)]-\rho h\omega^2 w=0 \tag{1}$$

where w is the modal function, E is Young's modulus, ρ is mass density, ω is the frequency of oscillation, and h is the beam thickness. It is assumed that the height of the nonuniform beam shown in Fig. 1 varies only slightly from some reference uniform beam of height h_0, so that

$$h(x)=h_0+\epsilon h_1(x) \tag{2}$$

where ϵ is a parameter here assumed to be less than unity, giving a measure of the variation from the reference uniform beam, and max h_1 is equal to or less than h_0.

The fundamental frequency and mode shape of the beam with variable thickness not differing too greatly from the uniform reference beam are represented as[4]

$$\omega^2=\omega_0^2+\sum_{i=1}^{I}\epsilon^i\omega_i^2 \tag{3}$$

and

$$w(x)=w_0(x)+\sum_{i=1}^{I}\epsilon^i w_i(x) \tag{4}$$

where ω_0^2 and $w_0(x)$ are the fundamental frequency squared and mode shape, respectively, of the reference beam with uniform thickness h_0. Equations (2-4) are substituted into Eq. (1), which, after neglecting terms of the order of ϵ^3 and higher and equating terms of equal powers of ϵ, yields the following perturbation equations.

Zeroth-order Perturbation Equation

$$EI_0\mathrm{d}^4w_0/\mathrm{d}x^4-\rho h_0\omega_0{}^2w_0=0 \text{ where } I_0=h_0{}^3/12 \tag{5}$$

First-order Perturbation Equation

$$EI_0\frac{\mathrm{d}^4w_1}{\mathrm{d}x^4}-\rho h_0\omega_0^2w_1$$
$$=-\frac{Eh_0^2}{4}\left[h_1\frac{\mathrm{d}^4w_0}{\mathrm{d}x^4}+\frac{2\mathrm{d}h_1}{\mathrm{d}x}\frac{\mathrm{d}^3w_0}{\mathrm{d}x^3}+\frac{\mathrm{d}^2h_1}{\mathrm{d}x^2}\frac{\mathrm{d}^2w_0}{\mathrm{d}x^2}\right]$$
$$+\rho[\omega_0^2h_1w_0+\omega_1^2h_0w_0] \tag{6}$$

Second-order Perturbation Equation

$$EI_0\frac{\mathrm{d}^4w_2}{\mathrm{d}x^4}-\rho h_0\omega_0^2w_2$$
$$=-\frac{Eh_0^2}{4}\left[h_1\frac{\mathrm{d}^4w_1}{\mathrm{d}x^4}+\frac{2\mathrm{d}h_1}{\mathrm{d}x}\frac{\mathrm{d}^3w_1}{\mathrm{d}x^3}+\frac{\mathrm{d}^2h_1}{\mathrm{d}x^2}\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}\right]$$
$$-\frac{Eh_0^2}{4}\left[h_1^2\frac{\mathrm{d}^4w_0}{\mathrm{d}x^4}+2\left(\frac{\mathrm{d}h_1}{\mathrm{d}x}\right)^2\frac{\mathrm{d}^3w_0}{\mathrm{d}x^3}+\left(\frac{\mathrm{d}^2h_1}{\mathrm{d}x^2}\right)^2\frac{\mathrm{d}^2w_0}{\mathrm{d}x^2}\right]$$
$$+\rho[\omega_2^2h_0w_0+\omega_1^2h_1w_0+\omega_0^2h_1w_1+\omega_1^2h_0w_1] \tag{7}$$

The zeroth-order perturbation equation (5) is identical to the differential equation describing free fundamental motion of a beam with height h_0. Thus ω_0 and w_0 can be determined for a given set of support conditions. With ω_0^2 and w_0 known, then ω_1^2 and w_1 can be found from Eq. (6) and the process continued to determine ω_2^2 and w_2 from Eq. (7). Assume now that the beam is simply supported with length ℓ. The exact solution to Eq. (5) for the fundamental mode is well known. To find a general solution to the first- and second-order perturbation equations (6) and (7), the perturbated modal functions $w_i(x)$ in Eq. (4) are represented as a sum of the eigenfunctions of the simply supported uniform beam

$$w_i(x)=\sum_{j=1}^{J}w_{ij}\sin(j\pi x/\ell) \tag{8}$$

The expressions for w_0 and w_i are then substituted into Eq. (6) and the resulting equation multiplied by one of the eigenfunctions and integrated over the length of the beam. One then obtains

$$\omega_1^2=\frac{2h_0}{\rho\ell}\int_0^\ell\left\{\frac{Eh_0^2}{4}\right.$$
$$\times\left[h_1\frac{\pi^4}{\ell^4}\sin\frac{\pi x}{\ell}-\frac{2\pi^3}{\ell^3}\frac{\mathrm{d}h_1}{\mathrm{d}x}\cos\frac{\pi x}{\ell}-\frac{\pi^2}{\ell^2}\frac{\mathrm{d}^2h_1}{\mathrm{d}x^2}\sin\frac{\pi x}{\ell}\right]$$
$$\left.-\rho\omega_0^2h_1\sin\frac{\pi x}{\ell}\right\}\sin\frac{\pi x}{\ell}\mathrm{d}x \tag{9}$$

The relative amplitudes of the coefficients in Eq. (8) can also be determined as

$$\frac{w_{1k}}{w_{01}}=\frac{-2h_0}{\rho_0\ell\pi^4k^4}\int_0^\ell\left\{\frac{Eh_0^2}{4}\right.$$
$$\times\left[h_1\frac{\pi^4}{\ell^4}\sin\frac{\pi x}{\ell}-\frac{2\pi^3}{\ell^3}\frac{\mathrm{d}h_1}{\mathrm{d}x}\cos\frac{\pi x}{\ell}-\frac{\pi^2}{\ell^2}\frac{\mathrm{d}^2h_1}{\mathrm{d}x^2}\sin\frac{\pi x}{\ell}\right]$$
$$\left.-\rho\omega_0^4h_1\sin\frac{\pi_x}{\ell}\right\}\sin\frac{k\pi x}{\ell}\mathrm{d}x\quad(k\neq 1) \tag{10}$$

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The development for the second-order correction is obtained from Eq. (7) in similar manner but the development is not included here.

B. Free Vibration of Square Plates

The differential equation describing the free transverse motion of a square thin plate with variable thickness is

$$\nabla^2(D\nabla^2 w)-(1-\upsilon)\left[\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-\frac{2\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]-\omega^2\rho h w=0 \tag{11}$$

where ν is Poisson's ratio and $D(x,y)$ is the plate flexural rigidity. Again, the thickness variation is assumed to be only slightly different from some reference uniform plate as given in Eq. (2). The frequency and mode shape of the plate with variable thickness are given by Eqs. (3) and (4) which represent only slight variations of the fundamental frequency and mode shape from the reference uniform plate. Equations (2-4) are substituted into Eq. (11) and terms of the same power of ϵ are equated to obtain the following set of differential equations

Zeroth-order Perturbation Equation

$$D^1 h_0^3\nabla^4 w_0-\rho\omega_0^2 h_0 w_0=0 \tag{12}$$

First-order Perturbation Equation

$$D^1 h_0^3\nabla^4 w_1-\rho\omega_0^2 h_0 w_1=-D^1 h_0^2\Big\{3\nabla^2 h_1\nabla^2 w_0+3h_1\nabla^4 w_0+6\frac{\partial h_1}{\partial x}\frac{\partial}{\partial x}(\nabla^2 w_0)+6\frac{\partial h_1}{\partial y}\frac{\partial}{\partial y}(\nabla^2 w_0)-(1-\upsilon)\Big[3\frac{\partial^2 h_1}{\partial y^2}\frac{\partial^2 w_0}{\partial x^2}$$
$$+3\frac{\partial^2 h_1}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}-6\frac{\partial^2 h_1}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y}\Big]\Big\}+\rho\{\omega_0^2 h_1 w_0+\omega_1^2 h_0 w_0\} \tag{13}$$

Second-order Perturbation Equation

$$D^1 h_0^3\nabla^4 w_2-\rho_0\omega_0^2 h_0 w_2=-D^1\Big\{3h_0^2\nabla^2 h_1\nabla^2 w_1+3h_0\nabla^2 h_1^2\nabla^2 w_0+3h_0^2 h_1\nabla^4 w_1+3h_0 h_1^2\nabla^4 w_0+6h_0^2\frac{\partial}{\partial x}(\nabla^2 w_1)$$
$$+6h_0\frac{\partial}{\partial x}h_1^2\frac{\partial}{\partial x}(\nabla^2 w_0)+6h_0\frac{\partial}{\partial y}h_1^2\frac{\partial}{\partial y}(\nabla^2 w_0)-(1-\upsilon)\Big[3h_0^2\frac{\partial^2}{\partial y^2}h_1\frac{\partial^2}{\partial x^2}w_1+3h_0\frac{\partial^2}{\partial y^2}h_1^2\frac{\partial^2}{\partial x^2}w_0+3h_0^2\frac{\partial^2}{\partial x^2}h_1\frac{\partial^2}{\partial y^2}w_1$$
$$+3h_0\frac{\partial^2}{\partial x^2}h_1^2\frac{\partial^2}{\partial y^2}w_0-6h_0^2\frac{\partial^2 h_1}{\partial x\partial y}\frac{\partial^2 w_1}{\partial x\partial y}-6h_0\frac{\partial^2 h_1}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y}\Big]\Big\}+\rho\{\omega_1^2 h_0 w_1+\omega_1^2 h_1 w_0+\omega_2^2 h_0 w_0\} \tag{14}$$

The fundamental mode shape of the plate with variable thickness is represented by a series of eigenfunctions of the reference uniform plate

$$w_i(x,y)=\sum_{m=1}^{M}\sum_{n=1}^{N} w_{i_{mn}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} \tag{15}$$

Expressions for w_0 and w_1 are substituted into Eq. (13), which is satisfied in the Fourier sense to obtain a first-order correction for frequency.

$$\omega_1^2=\frac{4}{\rho h_0 a^2}\int_0^a\int_0^a\Big\{\Big[-3(1+\upsilon)D^1 h_0^2\Big(\frac{\pi}{a}\Big)^2\nabla h_1+12D^1 h_0^2\Big(\frac{\pi}{\mathrm{a}}\Big)^2 h_1-\rho\omega_0^2 h_1\Big]\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}$$
$$-12D^1 h_0^2\Big(\frac{\pi}{a}\Big)^3\Big[\frac{\partial h_1}{\partial x}\cos\frac{\pi x}{a}\sin\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\ \frac{\partial h_1}{\partial y}\sin^2\frac{\pi x}{a}\cos\frac{\pi y}{a}\sin\frac{\pi y}{a}\Big]$$
$$+6(1-\upsilon)D^1 h_0^2\Big(\frac{\pi}{a}\Big)^2\frac{\partial^2 h_1}{\partial x\partial y}\cos\frac{\pi x}{a}\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\sin\frac{\pi y}{a}\Big\}\mathrm{d}x\mathrm{d}y \tag{16}$$

The magnitudes of $w_{1_{mn}}$ in Eq. (15) are

$$\frac{w_{1_{mn}}}{w_{0_{11}}}=\frac{-4a^2}{(m^4+2m^2n^2+n^4)\pi^4}\times\int_0^a\int_0^a\Big\{\Big[-3(1+\upsilon)\Big(\frac{\pi}{a}\Big)^2\nabla^2 h_1+12\Big(\frac{\pi}{a}\Big)^4 h_1-\frac{\rho}{D^1 h_0^2}\omega_0^2 h_1\Big]\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$
$$-12\Big(\frac{\pi}{a}\Big)^3\Big[\frac{\partial h_1}{\partial x}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}+\frac{\partial h_1}{\partial y}\sin-6(1-\upsilon)\Big(\frac{\pi}{a}\Big)^2\frac{\partial^2 h_1}{\partial x\partial y}\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\Big\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\mathrm{d}y\mathrm{d}x \tag{17}$$

The second-order correction to frequency and mode shape is obtained in a similar manner from Eq. (14), but is not presented in detail here.

Table 1 Fundamental frequency of a tapered beam and a plate tapered in one direction

Taper ratio, α	*Beam* Fundamental frequency $\omega(m\ell/EI_0)^{1/2}$ Perturbation	Ref. 2	% Difference
.5	7.11	7.12	0.2
.67	5.97	6.06	1.5
.75	5.36	5.34	0.3
.90	3.99	3.88	2.8

Taper ratio, α	*Plate* Fundamental frequency $\omega^2(\rho h_0 a^4/D_0)$ Perturbation	Ref. 3	% Difference
.2	748.2	748.2	0.00
.4	649.4	650.1	0.11
.6	556.6	557.2	0.10
.8	469.7	470.5	0.17

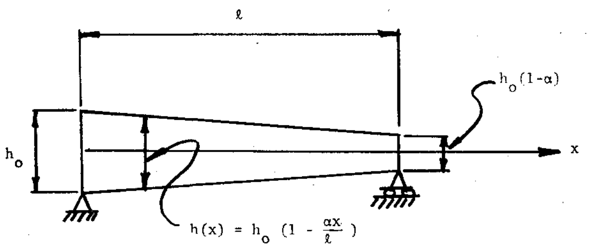

Fig. 1 Linearly tapered beam or cross section of a plate linearly tapered in the x direction.

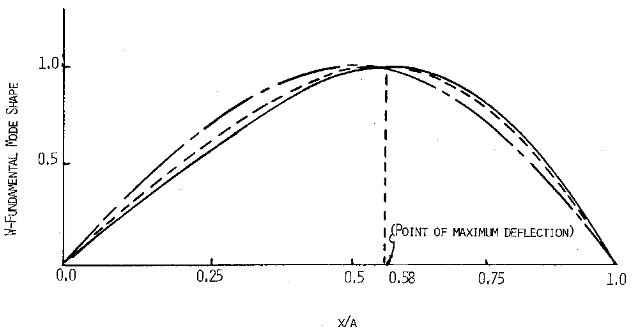

Fig. 2 Fundamental mode shape of a square plate linearly tapered in one direction (taper ratio $\alpha=0.8$, $y=a/2$). Legend: (–) perturbation; (----) Ref. 3; (– – –) plate with uniform thickness.

III. Examples

A. Linearly Tapered Beam

The linearly tapered beam shown in Fig. 1 was selected as an example. The thickness variation can then be expressed as

$$\epsilon h_1(x) = -\alpha h_0 x/\ell; \quad \epsilon=\alpha; \quad h_1(x) = -h_0 x/\ell \tag{18}$$

The approximate fundamental frequency and mode shape are determined from Eqs. (9) and (10) and similar equations representing the second-order corrections. The change of fundamental frequency of the tapered beam is shown in Table 1 for four values of taper ratio and is compared with the exact frequency obtained from Ref. 2.

B. Linearly Tapered Plate

A simply supported square plate with linearly varying thickness in the x direction as shown in Fig. 1 was considered. The thickness of such a plate can be functionally represented as in Eq. (18). The approximate frequency for this particular plate with variable thickness was calculated from Eq. (16) and a similar equation representing a second-order correction. The frequency change for various values of taper ratio, α, is shown in Table 1 and compared with the frequency determined from Ref. 3. The fundamental mode shape as determined from Eq. (17) is shown in Fig. 2 and compared to the mode shape obtained from Ref. 3. The mode shape as determined from Eq. (17) is

$$w = [\sin(\pi x/a) + .1784 \sin(2\pi x/a) + .004 \sin(4\pi x/a) + .0005 \sin(6\pi x/a)] \sin(\pi y/a) \tag{19}$$

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Two-Dimensional Radiative Equilibrium: A Simple Nongray Problem

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A RECENT review[1] of the literature on two-dimensional radiative equilibrium reveals that all of the analyses use the gray approximation. The spectra of many substances, such as glass, carbon monoxide, and water vapor have windows or regions in which the absorption coefficient is zero or very small. The inability of the gray model to account for these windows is one of its major limitations. The present study of nongray radiative transfer in a two-dimensional

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